Performance assessment of antithetic random fields in a stochastic mine planning model

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ABSTRACT: Conventional mine planning often relies on parameters estimation to obtain a single production plan. There is no guarantee that these estimations will be accurate in the long term, and this could lead to issues in the mine operation. To deal with this uncertainty, different optimization models have been proposed, which incorporate equally probable scenarios. Unfortunately, the incorporation of uncertainty also imposes a computational challenge: a large number of scenarios is desirable to capture the variability of the uncertain parameters, but each additional scenario increases the computational complexity of the mathematical problem, limiting the cases that can be addressed with stochastic optimization. This paper implements a variance reduction technique in the sequential Gaussian simulation algorithm, which generates grade scenarios with negative correlation, so fewer scenarios can be used without compromising the representation of the grade variability. Our experiments show that using these scenarios achieves the same precision in the objective value of a stochastic optimization problem, but using fewer simulations in the formulation compared with the conventional gaussian algorithm.

Keywords: Antithetic Random Fields, Variance Reduction, Stochastic Optimization, Mine Planning, Geoestatistics

1 INTRODUCTION

Uncertainty in Mine Planning has been a widely studied topic in the last decade. Several works have stated the effect of uncertainty in production plans, and the difficulty in achieving the value predicted based on estimated models (Smith & Dimitrakopoulos 1999 and Dimitrakopoulos et al. 2002). Given this scenario, different methodologies has been proposed to incorporate the uncertainty into the decision making process, using probable future scenarios for the uncertain parameters (costs, prices, grades, etc.). Among these methodologies, several stochastic optimization models has been proposed in the literature, which aim to obtain a production plan considering many scenarios in the formulation in order to obtain a more profitable or robust production schedule. Regarding to geological uncertainty, geostatistics provides simulation techniques, which generates several possible geological scenarios, which can be incorporated into stochastic optimization models. However, the optimization models related to mine planning are often difficult to solve, and this issue is accentuated in the stochastic framework with several scenarios. This poses a clear trade-off: a high number of scenarios is needed to ensure a good representation of the true variability of the uncertain parameter. At the same time, each scenario makes the problem harder to solve, which is a limiting factor to address complex and large mines. Given this scenario, in this paper we evaluate the performance of the antithetic random fields technique, which aims to reduce the number of grade scenarios needed in a optimization problem, without compromising the representation of the true grade variability, based on generating scenarios with negative pairwise correlation. This technique is tested in a particular stochastic framework, which generates a single production plan based on multiple scenarios.

Scenario reduction has been studied in the last decades with different approaches. One of them is based on selecting a small set that represents the characteristics of a larger set. Usually, the size of the small set is fixed beforehand, and the objective is finding the best subset that minimizes the difference between them considering a relevant attribute for a particular problem. This has been used by Heitsch & Römisch (2003), where they proposed a heuristic to find such subset, defining the relevant distance and adding or subtracting a single scenario aiming to minimize it. A similar methodology was used by Dupačov et al. (2003) and Heitsch & Römisch (2009) in power management. Armstrong et al. (2013) proposed a random search procedure to find a representative subset in an stochastic mine planning problem. They used a proxy variable (ore tonnes above certain cut-off grades) to characterize the differences between scenarios, and the random search procedure proved its usefulness reducing the number of scenarios needed for the stochastic formulation in a real deposit. A similar approach, using proxy variables related to the expected response of a transfer function, has been used by Deutsch & Srinivasan (1996) and McLennan & Deutsch (2005). These proxy variables are used to construct a ranking of realizations to then select a small subset to characterize the uncertainty on the response. This allows to process only a subset of the complete scenarios, typically linked to some key percentiles to represent the variability in the response (Deutsch 2007 and Pereira et al. 2017).

Another approach has been the use of variance reduction techniques, which are procedures to reduce the variance on some estimation without increasing the number of realizations, based on modifications on the algorithm used to generate the scenarios. There are many techniques under this category, and more details can be found in Cheng (1986) and James (1985). In this work, we will use a variance reduction technique known as antithetic variates (Hammersley & Morton 1956), which is based on generating scenarios with negative correlation, expecting that maximizing the difference among the scenarios allows a better representation of the expected variability of the uncertain parameter. An extension of this methodology to a sequential simulation algorithm was proposed by Guthke & Bárdossy (2012). This extension allows to generate an arbitrary number of scenarios negatively correlated in a geostatistical framework. This was used in Nelis et al. (2018) in an stochastic mine planning problem, which achieved a significant variance reduction in the NPV estimation of the mine schedule. In this work, we will use this variance reduction technique in a real case study, using a widely studied stochastic framework, based on the minimization of the deviations from the production targets considering multiple grade scenarios. This framework was first proposed as an optimization problem in Ramazan & Dimitrakopoulos (2007), but many extensions has been proposed such as multiple elements (Benndorf & Dimitrakopoulos 2013), different processing streams (de Freitas et al. 2015) and mining complexes (Goodfellow & Dimitrakopoulos 2016). Specifically, the performance of the antithetic random fields technique will be tested in an optimization model based on Leite & Dimitrakopoulos (2014). This optimization model, along with the antithetic random fields algorithm will be detailed in the next section.

2 METHODS

2.1 *Antithetic Random Fields*

The antithetic random fields technique is based on the work of Guthke & Bárdossy (2012). They propose a methodology to generate an arbitrary number of scenarios with negative correlation in a sequential simulation algorithm, modifying the generation of the random numbers needed in each scenario. For a conventional sequential Gaussian simulation, each node in the simulation is visited in a random order, and a random number is generated to obtain the simulated value for that node. A new random path is generated for the following scenarios, and the same node is simulated using a different, iid random number. In the algorithm proposed by Guthke & Bárdossy (2012), the same random path is used for all the scenarios, and the random numbers for each node are drawn beforehand. Moreover, a correlation matrix is imposed to these random numbers to obtain a negative pairwise correlation among every scenario. Then, the negative correlation of the random number transfers to the complete simulated scenarios.

Formally, let m be the number of simulated scenarios required, and n the number of nodes in each scenario. To simulate these m scenarios, m standard Gaussian vectors of size n are needed, one vector for each scenario and one element of these vectors for each node. To achieve the negative correlation between scenarios, a correlation matrix is imposed to these m vectors.

Considering Equation (1) as the correlation matrix with α the correlation coefficient. Equation (2) establishes the dependence between the number of scenarios and the correlation coefficient (Guthke & Bárdossy 2012).

$$\mathbf{C}_m = \begin{pmatrix} 1 & \alpha & \cdots & \alpha \\ \alpha & 1 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & 1 \end{pmatrix} \tag{1}$$

$$\alpha \geq -\frac{1}{m-1} \tag{2}$$

To impose this correlation matrix between m random vectors, the following algorithm must be followed:

1. Construct Matrix $\mathbf{C}_m$
2. Decompose this matrix in $\mathbf{C}_m = \mathbf{B}\mathbf{B}^\intercal$
3. For each node i to simulate:
   i. A tuple of size m with standard random Gaussian numbers is generated, $\mathbf{g}_m^i$
   ii. Impose the correlation coefficient as $\mathbf{z}_m^i = \mathbf{B}\mathbf{g}_m^i$
4. The collection of vectors $\mathbf{z}_m^i$ form the matrix $\mathbf{R} = (\mathbf{z}_m^i)^\intercal$. Each column of this matrix is a Gaussian random vector of size n, and each row is a tuple of size m. The correlation matrix among these m vectors is $\mathbf{C}_m$

Then, the sequential Gaussian algorithm is modified to use the same random path for each scenario in the m-tuple and the corresponding random number from matrix $\mathbf{R}$, which allows to generate negatively-correlated scenarios.

For each m-tuple of scenarios:

1. **Random Path:** A random path is generated.
2. **Random Numbers:** Matrix $\mathbf{R}$ is constructed.
3. For each scenario s in the m-tuple:
   i. **Simple Kriging:** Visit each node i from scenario s according to the random path and perform a simple kriging estimation using nearby data and any previously simulated nodes.
   ii. **Simulate Value:** Assign the value of this node as:

$$Y(x_i) = Y^{KS}(x_i) + \sigma^{KS}(x_i)\mathbf{R}_{i,s} \tag{3}$$

The scenarios obtained by this method are used to solve the optimization problem described in section 2.2, using the convergence analysis described in section 2.3. Notice that these Gaussian random fields are back-transformed to match the original grade histogram.

2.2 *Minimization of deviations*

The optimization model used to evaluate the performance of the antithetic random fields technique is presented in this section. This model is based on Leite & Dimitrakopoulos (2014), and aims to obtain a single production plan maximizing the expected value of the extraction and, simultaneously, maintaining a consistent production target for every scenario. A cut-off grade is used to classify each block as ore or waste, and since each scenario presents different grades, the same block may have a different classification. Therefore, for the same mine sequence, some scenarios may have a surplus of ore or metal content from the desired target, meanwhile others may have a shortage, which are called deviations. Since the excess and shortage of ore is detrimental for the mining operation, the optimization model incorporates a cost in the objective function accounting for the deviations from the production targets of each scenario. Therefore, the model aims to minimize this cost finding a feasible schedule for all the scenarios at the same time.

Formally, the optimization model is defined as follows: let $\mathcal{B}$ be the set of mining blocks, $\mathcal{T}$ the set of periods of the schedule, $\mathcal{R}$ the set of resources consumed in the extraction process and $\mathcal{S}$ the set of grade scenarios. v_{bt} is the expected profit obtained if block $b \in \mathcal{B}$ is extracted at period $t \in \mathcal{T}$. Each resource $r \in \mathcal{R}$ at period $t \in \mathcal{T}$ has an upper and lower limit which define the production targets, denoted as U_{rt} and L_{rt} respectively. Associated to these targets, parameters c_r^u

and c_r^l are the surplus and shortage deviation cost per unit of tonne of resource $r \in \mathcal{R}$ considering that each block $b \in \mathcal{B}$ has a resource attribute of r_{bs} at scenario $s \in \mathcal{S}$. This resource can be denoted as r_b as well, when it is the same amount for all the scenarios. A geological discount rate, ρ_g, controls the cost of deviations through the planning horizon: deviations in an earlier period will have a higher cost than deviations in later periods, which aims to extract blocks with low uncertainty at the beginning of the schedule. Finally, each block $i \in \mathcal{B}$ has a precedence set $\mathcal{P}(i)$ which contains all the blocks that must be extracted before allowing the extraction of block i. The decision variables for this model are defined as follows:

$$x_{bt} = \begin{cases} 1 & \text{if block } b \in \mathcal{B} \text{ is extracted at period } t \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$d_{str}^u = \text{Surplus from production target } U_{rt} \text{ at period } t \in \mathcal{T} \text{ in scenario } s \in \mathcal{S} \tag{5}$$

$$d_{str}^l = \text{Shortage from production target } L_{rt} \text{ at period } t \in \mathcal{T} \text{ in scenario } s \in \mathcal{S} \tag{6}$$

Using these definitions, the stochastic mine planning model to minimize the deviations is defined as:

$$\max \quad \sum_{b\in\mathcal{B}}\sum_{t\in\mathcal{T}} x_{bt}\,\bar{v}_{bt} - \frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}\sum_{t\in\mathcal{T}}\sum_{r\in\mathcal{R}}\rho_g(d_{str}^u\,c_r^u + d_{str}^l\,c_r^l) \tag{7}$$

$$\sum_{b\in\mathcal{B}} x_{bt}\,r_{bs} + d_{str}^l \geq L_{rt} \qquad \forall s \in \mathcal{S}, r \in \mathcal{R}, t \in \mathcal{T} \tag{8}$$

$$\sum_{b\in\mathcal{B}} x_{bt} - d_{str}^u \geq U_{rt} \qquad \forall s \in \mathcal{S}, r \in \mathcal{R}, t \in \mathcal{T} \tag{9}$$

$$L_{rt} \leq \sum_{b\in\mathcal{B}} x_{bt}\,r_b \leq U_{rt} \qquad \forall t \in \mathcal{T} r \in \mathcal{R} \tag{10}$$

$$x_{it} \leq \sum_{p=1}^{t} x_{jp} \qquad \forall t \in \mathcal{T}, j \in \mathcal{P}(i) \tag{11}$$

$$\sum_{t\in\mathcal{T}} x_{bt} \leq 1 \qquad \forall b \in \mathcal{B} \tag{12}$$

$$d_{str}^l \geq 0,\; d_{str}^u \geq 0 \qquad \forall s \in \mathcal{S}, r \in \mathcal{R}, t \in \mathcal{T} \tag{13}$$

Equation (7) is the objective function. The first term is related to the expected NPV maximization of the schedule, and the second term is the total cost of deviations for every resource considered and for every geological scenario. The base formulation in Leite & Dimitrakopoulos (2014) does not normalize this term by the total number of simulations, but this modification allows to compare the objective function value among instances with a different number of geological scenarios, which is a fundamental part of the convergence analysis. Equation (8) is the Lower Resource Deviation constraint, which accounts for the deviation from lower production target at scenario $s \in \mathcal{S}$. Since the same block may present a different amount of ore or waste tonnes in different scenarios, each simulation is associated with a different deviation constraint. Similarly, Equation (9) accounts for the excess of production of resource r from the upper production target in each scenario. Since not every resource is allowed to deviate from the production target, Equation (10) represents the capacity constraint for such resources. Equation (11) represents the precedences constraints, which establish a spatial order of the extraction to maintain the pit stability. Equation (12) states that each block can be extracted once. Finally, Equation (13) establishes the bounds for the deviations variables.

2.3 *Convergence Analysis*

The performance of the antithetic random fields technique (Section 2.1) in the stochastic mine planning problem (Section 2.2) is evaluated with the following methodology. An *instance* of problem (7) is the set of parameters used to define the objective function and the constraints

for a particular case study, such as prices, costs, attributes, etc. More relevant for this study, an instance is defined for the set of scenarios used to represent the grade variability. If the number of scenarios, $|S|$, is a good representation of this variability, every instance of problem (7) should have a similar optimal objective function value. Equivalently, the dispersion in the objective function value among different instances with the same sample size $|S|$ gives information about the representation of the grade variability for that sample size. Therefore, the evaluation is based on defining several instances of problem (7) with different sample size and different simulation algorithms, and studying their dispersion. The comparison between the results obtained with different simulation algorithms is an indicator of the performance of such algorithms representing the true grade variability of the deposit.

For this case study, the procedure to compare the performance of conventional and antithetic simulations is described as follows:

1. **Scenario generation:** The scenarios necessary to define the instances are generated according to each simulation algorithm:
   i. For the conventional sequential Gaussian algorithm, 600 independent scenarios are generated
   ii. For the antithetic scenarios, 600 realizations are generated in total, but given the nature of this algorithm they are generated in negative-correlated sets of size m. Firstly, 300 sets of $m = 2$ are generated ($\alpha = -1$), which are noted ARF2. Then, 60 sets of $m = 10$ are generated ($\alpha = -0.\overline{11}$), which are noted ARF10
2. **Instance definition:** Using the previously generated scenarios, 30 instances are defined for each simulation algorithm and sample size. The sample sizes $|S|$ chosen for this analysis were 2, 10 and 20. Depending on the simulation algorithm, the definition of these instances is different, which is detailed next:
   i. **Conventional simulations:** 30 sets of $|S| = 2$ are randomly selected from the pool of scenarios, each scenario picked individually. Equivalently, 30 sets of $|S| = 10$ and 30 sets of $|S| = 20$ are selected randomly.
   ii. **ARF2:** For the sample size $|S| = 2$, 30 sets of paired realizations are selected. For $|S| = 10$, each one of the 30 instances is made of 5 sets of ARF2, each set picked randomly. Finally, for $|S| = 20$, 30 instances are defined, each one with 10 sets of ARF2.
   iii. **ARF10:** For $|S| = 2$, 30 pairs of realizations are picked randomly, each one of these pairs is selected from a single set of ARF10. For $|S| = 10$, each instance is defined with a single set of ARF10. Similarly, for $|S| = 20$, each one of the 30 instances is defined by 2 sets of ARF10 picked randomly.
3. **Statistical analysis:** For each case, the 30 instances are solved and the optimal values of the objective function is obtained. The mean value of these 30 instances is an estimator of the true objective function value, while the standard deviation is a measure of the precision of this estimation for each sample size and simulation algorithm. The performance of this antithetic random fields technique is evaluated comparing the standard deviation obtained by using ARF2 and ARF10, and the standard deviation of the conventional simulations.

3 RESULTS

3.1 *Case study*

Drillhole data from a deposit located in Northern Chile were used. The element of interest is copper, with an average grade of 0.28%. The drillhole data cover a total area of 2.0×2.8 km^2 in a pseudo-regular grid of 100m $\times$ 100m. For the case study, the central part of the deposit was considered, in a zone of 800×800 m^2. Most of those drillholes are 200m depth given the mantle disposition of the orebody. General statistics about the data can be found in Table 1

The normal score of the drillhole data was used to perform the point-support Gaussian simulation in a regular grid of $216 \times 216 \times 28$ nodes, with a separation of $3.75\text{m} \times 3.75\text{m} \times 7.5\text{m}$ between them. A change of support is performed latter, to obtain the final block model with 20412 blocks of $15 \times 15 \times 15$ m^3.

Regarding the optimization model, for each scenario, a cut-off grade was used to calculate the block profit of each block, and later the average of these profits was used as the expected

Table 1. Basic statistics for the copper content

Parameter	Value
Mean	0.28 %
Maximum	2.74 %
Minimum	0.09 %
Standard Deviation	0.17 %
Data points	15,622

value in the optimization model. Since this calculation can be done prior to the optimization process, 600 scenarios were used for this estimation. For the deviations of each instance, the number of scenarios is variable according to the sample size selection described in section 2.3. The mining capacity is fixed for every scenario and does not allow deviations. On the other hand, the processing capacity allows surplus and shortage deviations from the production target associated with the ore tonnes of each scenario. The scheduling parameters for the optimization model can be found in Table 2.

Table 2. Scheduling parameters for the optimization model

Operational		Economic	
Parameter	Value	Parameter	Value
Planning horizon	5 years	Price	2.5 USD/lb
Mine Capacity	30 MTon	Mining Cost	1.0 USD/Ton
Upper Processing Target	28 MTon	Processing Cost	10 USD/Ton
Lower Processing Target	28 MTon	Selling Cost	0.5 USD/lb
Slope Angle	45°	Recovery	90 %
		Economic Discount Rate	10%
		Geological Discount Rate	10%
		Upper Deviation Cost	20 USD/Ton
		Lower Deviation Cost	20 USD/Ton

With these parameters, the methodology proposed in section 2 was implemented, and the results of the dispersion of the NPV values will be presented in the next section.

3.2 *Scheduling Results*

The dispersion of objective function value for each simulation algorithm and sample size is shown in Figure 1. The middle line in each boxplot represents the average objective function value of 30 instances for each simulation and sample size, while the size of the box is equivalent to two times the standard deviation of these values, to represent the dispersion obtained for each case. Finally, the upper and lower whiskers show the maximum and minimum value of the objective function.

The first main result is that, for all the types of simulation considered, the larger the sample size $|S|$, the lower the variance of the instances. This result matches the expectation of an improvement in the accuracy on the estimation when a larger sample is taken into account since the true variability of the deposit is represented in a better way. The magnitude of this reduction depends on the type of simulation: Conventional Gaussian simulation achieves a reduction in the standard deviation of 55% from a sample of $|S| = 2$ to a sample of $|S| = 10$, and a reduction of 40% from a sample of $|S| = 10$ to $|S| = 20$. For $ARF2$, the reduction is 56% from a sample of $|S| = 2$ to $|S| = 10$, and 31% from $|S| = 10$ to $|S| = 20$. Finally, for ARF10, the results show an abnormal behavior: a reduction of 69% in the standard deviation is achieved from $|S| = 2$ to $|S = 10$, which is the highest reduction in this case study, but there is an increase of 4% in the standard deviation from $|S| = 10$ to $|S| = 20$.

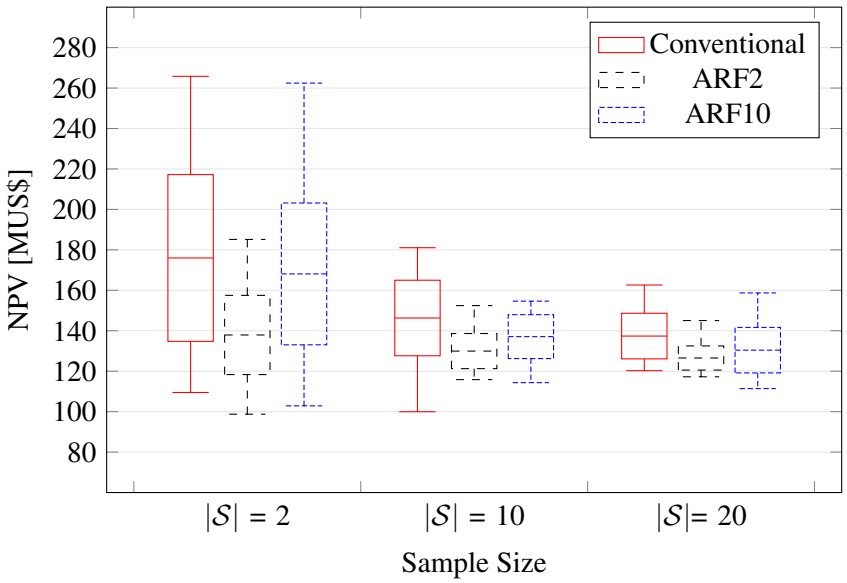

Figure 1. Dispersion of the objective function value

This unexpected behavior is explained by the correlation for different sample sizes. In the antithetic random fields technique, the highest the n-tuple to be negatively correlated, the weaker the pairwise negative correlation between the elements of the tuple. For this case, the average pairwise correlation of the elements in a 10-tuple of ARF10 is 0.47. This value is similar to the average pairwise correlation in conventional simulation for this study case, which is 0.52. Therefore, when the sample size is $|S| = 2$, the effect of the negative correlation is weak and a mild reduction in the standard deviation compared with the conventional simulation is presented. This changes with a sample size of $|S| = 10$, where there is a match between the tuple size and the sample size, and therefore the negative correlation allows a good representation with this sample size, explaining the reduction of 69% both for the larger sample size and for the match between sample size and tuple size. This behavior is not present with $|S| = 20$ where the sample size does not match the tuple size and the standard deviation is similar to the conventional case.

Comparing among the different simulation types at the same sample size, the results show that ARF2 achieves the lowest standard deviation for every sample size, even when the sample size does not match the tuple size. This is explained given the strong pairwise antithetic correlation with a tuple of size 2, which for this case study is 0.06. This allows a good representation of the deposit variability even with a sample size of $|S| = 2$, which allows a standard deviation reduction of 52% compared to the conventional simulation algorithm and 44% compared to ARF10, which is similar to the conventional case as it was discussed previously. For a sample size of $|S| = 10$, the tuple does not match the sample size for ARF2, but the average pairwise correlation coefficient of the elements of each sample is 0.38, which is lower than both the conventional and ARF types of simulations, allowing a better sampling with the same number of scenarios. The reduction in the standard deviation for ARF2 in this sample size is 54% compared to the conventional algorithm and 20% compared to ARF10. Finally, for a sample size of $|S| = 20$, ARF2 achieves the lower standard deviation as well, with a reduction of 47% compared to both conventional and ARF10 simulations.

A possible explanation for these results is related to the nature of the optimization problem. Since the value of the extraction is estimated prior to the simulation process, the geological scenarios are only used to estimate the deviations from the production targets. Since the deviations only consider the classification of the material as ore or waste, the differences among different scenarios are based on whether each block is above or below the cut-off grade. Therefore, a favorable way to sample for this optimization model is focusing on the cases when each block belong to each category, and not necessarily on the complete range of grades for each block. This could explain the performance of ARF2, compared with the rest of the simulation algorithms. Different case studies for this optimization model should be tested to confirm this behavior.

An unexpected result is the difference between the average values of the objective function for each type of simulation and sample size. For every sample size, the average value is lower for ARF2 and higher for the conventional simulation. Moreover, the average value tends to decrease when the sample size is larger. This behavior was not seen in previous works, where the average value was similar for every simulation type (Nelis et al. 2018). The apparent bias is larger with smaller sample sizes, where the average value for conventional algorithm is 22% higher than the average value of ARF2 with $|S| = 2$. This difference decreases with larger sample sizes: for $|S| = 10$, the difference is 11% between the conventional simulations and ARF2 and for $|S| = 20$ gets to 7%. The differences between ARF10 and Conventional are smaller: 4% for $|S| = 2$, 6% for $|S| = 10$ and 5% for $|S| = 20$. Since the true value of the objective function is unknown, it is not possible to establish whether the Antithetic Simulations present a bias. However, it can be seen that the difference between the average value among different simulation types gets smaller with larger sample sizes, so it is expected this difference in the expected value tends to zero with even larger sample sizes. This also could be an indication that even a sample size of $|S| = 20$ is not large enough to guarantee convergence for this case study.

4 CONCLUSIONS

The performance of a variance reduction technique was presented in this work. The results showed that the antithetic random fields implementation achieved a variance reduction in the estimation of the objective value of a particular long-term stochastic optimization model in a case study using real data. A difference between the expected value estimation using this variance reduction technique compared to conventional simulation algorithm was found, but it approached to zero with larger sample sizes. This variance reduction technique shows promising results, and could lead to solve larger cases with fewer scenarios.

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